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Regularity of solutions for spatially homogeneous Boltzmann equation without angular cutoff (non Maxwellian molecule type)

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1 Introduction

We consider the Cauchy problem for the spatially homogeneous Boltzmann equation without angular cutoff

$$\begin{aligned} &\partial_t f(t,v) = Q(f,f)(t,v), \quad t\in\mathbb{R}^+,\ v\in\mathbb{R}^3, \\ &f(0,v) = f_0(v), \end{aligned} \tag{1.1}$$

where $f(t,v)$ is the distribution function of particles at time t with velocity v. In this note we present the main result obtained in [12] that any weak solution to the problem (1.1) satisfying the natural boundedness on mass, energy and entropy (see, [21]), that is,

$$\sup_{0<t}\int_{\mathbb{R}^3} f(t,v)[1+|v|^2+\log(1+f(t,v))]dv < +\infty, \tag{1.2}$$

is in the Sobolev space $H^{+\infty}(\mathbb{R}^3)$ or even in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ for any $t>0$. There are extensive studies on this problem and some related results, see [11, 4, 5]. However, to our knowledge, this problem has not been completely solved in the sense that some extra conditions are assumed besides the natural bounds on mass, energy and entropy. The improvement made in [12] allows us to remove these extra conditions, by using pseudo-differential calculus developed in [16] (cf., [6, 7]).

As usual, the collision operator $Q(g,f)$ in (1.1) is a bi-linear functional representing the change rate of the particle distribution through elastic binary collisions, and it takes the form

$$Q(g,f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(|v-v_*|,\sigma)\{g(v'_*)f(v') - g(v_*)f(v)\}d\sigma dv_*, \tag{1.3}$$

and

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \tag{1.4}$$

which give the relations between the post and pre collisional velocities. The non-negative function $B(|z|,\sigma)$ called the Boltzmann collision cross section depends only on $|z|$ and the scalar product $\langle \frac{z}{|z|},\sigma\rangle$ for monatomic gas. We assume that

$$B(|v-v_*|,\cos\theta) = \Phi(|v-v_*|)b(\cos\theta), \tag{1.5}$$

where Φ and $b(\cos\theta)$ can take the following two forms corresponding to the modified (soft or Maxwellian or hard) potentials and the Debye-Yukawa potential. That is, either

$$\Phi(|v-v_*|) = (1+|v-v_*|^2)^{\frac{\gamma}{2}}, \qquad \gamma \le 1, \tag{1.6}$$

$$\sin\theta b(\cos\theta) \approx K\theta^{-1-\nu}, \qquad 0<\nu<2, \tag{1.7}$$

or

$$\Phi(|v-v_*|) = (1+|v-v_*|^2)^{\frac{1}{2}}, \tag{1.8}$$

$$\sin\theta b(\cos\theta) \approx K\theta^{-1}(\log\theta^{-1})^{\mu}, \text{ when } \theta\to 0+,\ \mu>0, \tag{1.9}$$

for some constants $K>0$.

Recall that the potential of the inverse power law $\frac{1}{\rho^s}, s>1$, ρ being the distance between two particles, has the form (1.5) where the kinetic factor related to the relative velocity is given by

$$\Phi(|v-v_*|) \approx |v-v_*|^{1-\frac{4}{s}},$$

and the factor related to the collision angle has the singularity,

$$\sin\theta b(\cos\theta) \approx \frac{K}{\theta^{1+\nu}} \text{ when } \theta\to 0,$$

for $0<\nu=\frac{2}{s}<2$ (see [10, 23], for example). The cases $1<s<4$, $s=4$ and $s>4$ correspond to so-called soft, Maxwellian and hard potentials respectively. Notice that the Boltzmann collision operator is not well-defined for the case $s=1$ which corresponds to the Coulomb potential. The form (1.5) corresponding to Debye-Yukawa potential was proposed in [16] for the first time, see its appendix.

The fact that $\sin\theta b(\cos\theta)$ has a non-integrable singularity around $\theta=0$ in the case (1.7) is usually removed by applying the Grad's angular cutoff assumption. This assumption has played an intrinsic role for the profound progress of the mathematical theories and phenomena investigations of the Boltzmann equation. On the other hand, it is now well established that the Boltzmann collision operator without angular cutoff behaves like a singular integral operator or pseudo-differential operator whose leading term is characterized by the operator $(-\Delta)^{\nu/2}$. This was first pointed out by Pao [17], see also Ukai [20] where the Boltzmann equation without angular cutoff was studied for the first time in Gevrey classes, and was formulated explicitly by Lions [14] based on the regularity properties of the collision gain term[13] (see also [9, 15, 25]). The optimal Sobolev exponent $\nu/2$ is due to Villani [22]. Around 2000s, the regularity induced by the grazing collision was analyzed in terms of the entropy production integral (, cf. the work [2] and others in its refereces). In particular, [2] establishes several elegant formulations associated with the collision operator which have been essentially used to the study of the spatially homogeneous problem.

It should be noted in our assumptions that the factor Φ in the cross section related to the relative velocity is modified by adding the constant 1 and this is why we call them modified potentials. By adding this constant, we avoid the degeneracy and singularity when $v=v_*$ so that the function $\Phi(z)$ is smooth and has a uniform positive lower bound. The similar modifications are also assumed in [11, 4, 5]. How to remove this artificial assumption rigorously is still not known.

Now, we can state our main results in [12]. The first result is concerned with the case when the angular singularity of the cross section is mild.

Theorem 1.1 *Suppose that the cross section B satisfies* (1.6)-(1.7) *for* $0 \le \gamma \le 1, 0 < \nu < 1$ *or* (1.8)-(1.9). *Let f be any weak solution satisfying* (1.2) *and the mass conservation. Then, f is in $H^{+\infty}(\mathbb{R}^3)$ for any $t > 0$, or more precisely,*

$$f \in L^\infty([t_0, T];\ H^{+\infty}(\mathbb{R}^3)),$$

for any $T > 0$ and $t_0 \in (0, T)$.

This theorem does not rely on the existence of L^1 moments, while the following theorem *does* depend on it essentially. Actually, we consider the weak solutions satisfying

$$|v|^m f \in L^\infty([T_0, T_1]; L^1(\mathbb{R}^3)), \tag{1.10}$$

for all $m \in \mathbb{N}$ and for some $0 \le T_0 < T_1$. Notice that $T_0 = 0$ means the propagation of moment while $T_0 > 0$ means the moment gain.

Theorem 1.2 *Let*[1] $\gamma \le 1$. *Suppose* (1.6)-(1.7) *for* $0 < \nu < 2$. *Let f be any weak solution satisfying* (1.2), *the mass conservation and the moment condition* (1.10) *for some* $0 \le T_0 < T_1$. *Then, f is in $\mathcal{S}(\mathbb{R}^3)$, or more precisely,*

$$f \in L^\infty([t_0, T_1];\ \mathcal{S}(\mathbb{R}^3)),$$

for any $t_0 \in (T_0, T_1)$.

We remark that the existence of weak solutions to the Cauchy problem (1.1) without angular cutoff has been proved by Villani [21], under the sole assumption that initial data have the finite mass, energy and entropy (see (1.2) and Definition 3.1 below). These solutions are called the entropy solutions.

One of the important properties of the entropy solutions for the hard potentials (namely $\gamma > 0$) is, according to the work by Wennberg [24] (cf., Bobylev[8]), the moment gain property. That is, the L^1 moments of arbitrary order are created as soon as $t > 0$ even if initial data do not have finite moments. It should be remarked that we do not know whether this moment gain property can be justified to all entropy solutions for the hard potentials, because the finiteness of moment is formally assumed to show the uniform estimate concerning the moment (see (59) in [8]). It is obvious that the entropy solutions constructed by [21] enjoy this moment gain property since they are obtained as limits of solutions for angular cutoff Boltzmann equations. The uniqueness of weak solution to homogeneous Boltzmann equation is still open problem except for Maxwellian molecule case, cf. [18, 19]. In Section 2, we give a new result concerning the uniqueness for soft potential case, see Theorem 2.2

There are at least two previous results [11, 5] closely related to Theorem 1.1 and 1.2. First of all, Desvillettes and Wennberg [11] stated that for the case of the angular non-cutoff and non-Maxwellian molecule, there exist weak solutions to (1.1) acquiring $\mathcal{S}$ regularity for $t > 0$. Actually, these authors constructed such weak solutions by solving the approximate problem

$$\begin{aligned} &(f_\epsilon)_t = Q(f_\epsilon, f_\epsilon) + \epsilon \Delta_v f_\epsilon, \\ &f_\epsilon(0, \cdot) = f_0 * \phi_\epsilon, \end{aligned}$$

and by taking the limit when ϵ tends to zero, where ϕ_ϵ is a sequence of mollifiers with $\epsilon > 0$. Notice again that the uniqueness of the weak solution is unknown. Also, notice that the proof uses in an essential way the result on the L^1 moment gain. On the other

[1] The case $0 \le \gamma < 1$ is only considered in [12], but the proof there is applicable to the case $\gamma < 0$.

hand, Alexandre and Safadi [5] successfully show that any entropy solution is in $\mathcal{S}$ for modified hard potentials in positive time. However, in their work, another assumption is introduced on the weak solutions, that is, the existence of L^2 moments of arbitrary order,

$$f(t,v) \in L^\infty([t_0,+\infty); L^2_r(\mathbb{R}^3)) \quad \text{for any} \quad r \in \mathbb{R}. \tag{1.11}$$

The proof of our theorems is largely based on some sharp estimates of commutators of the collision operators and pseudo-differential operators. The technique developed for it gives an improved upper estimate of the collision operator, such as those studied in [1, 3]. The next Section 2 is devoted to presenting this upper estimate, together with lower and commutators estimates, which have been refined and given newly in our recent joint work [7] with R.Alexandre and C.-J.Xu. In Section 3 we give a sketch of proofs of Theorems 1.1 and 1.2.

2 Upper and lower estimates for collision operator

We adopt the notations for the weighted function spaces,

$$\begin{aligned} &\|f\|_{L^p_r} = \|f(v)\langle v\rangle^r\|_{L^p}, \qquad 1 \le p \le \infty, \quad r \in \mathbb{R}, \\ &\text{and} \\ &\|f\|^2_{H^s_r} = \int_{\mathbb{R}^n} |\langle D\rangle^s \langle v\rangle^r f(v)|^2 dv, \qquad s, r \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where $\langle v\rangle = (1+|v|^2)^{\frac{1}{2}}$ and $\langle D\rangle$ is the pseudo-differential operator with the symbol $\langle \xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. We often write $\langle v\rangle^l = W_l$ for $l \in \mathbb{R}$.

Firstly we state the upper estimate of the non-cutoff collision operator.

Theorem 2.1 *Let the collision cross section B be of the form* (1.5) *satisfying* (1.6) *and* (1.7). *Then for any $m \in \mathbb{R}$, one has*

$$\|Q(f,\, g)\|_{H^m(\mathbb{R}^3_v)} \le C\|f\|_{L^1_{(\gamma+\nu)^+}(\mathbb{R}^3_v)}\|g\|_{H^{m+\nu}_{(\gamma+\nu)^+}(\mathbb{R}^3_v)}\,, \tag{2.2}$$

where $k^+ = \max(k, 0)$.

Remark 2.1 *Similar estimates are given by [1, 3], including the case of Besov space. However, the estimates there require the weighted Sobolev or Besov norm of f to estimate the left hand side.*

For the proof of (2.2) it suffices to show

$$\left|\Big(Q(f,\, g),\; h\Big)_{L^2(\mathbb{R}^3_v)}\right| \le C||f||_{L^1_{(\gamma+\nu)^+}(\mathbb{R}^3_v)}||g||_{H^{m+\nu}_{(\gamma+\nu)^+}(\mathbb{R}^3_v)}||h||_{H^{-m}(\mathbb{R}^3_v)}. \tag{2.3}$$

Our method for the proof of (2.3) leads us to a more general estimate

$$\begin{aligned} &\left|\Big(W_l Q(f,\, g),\; h\Big)_{L^2(\mathbb{R}^3_v)}\right| \\ &\le C||f||_{L^1_{l^+ + (\gamma+\nu)^+}(\mathbb{R}^3_v)}||g||_{H^{m+\nu}_{(l+\gamma+\nu)^+}(\mathbb{R}^3_v)}||h||_{H^{-m}(\mathbb{R}^3_v)}, \end{aligned} \tag{2.4}$$

where $l \in \mathbb{R}$. Hence we have

Corollary 2.1 *Let the cross section B be the same as in Theorem 2.1. Then for any $m, l \in \mathbb{R}$*

$$\|Q(f,\, g)\|_{H_l^m(\mathbb{R}_v^3)} \le C\|f\|_{L^1_{l^+ + (\gamma+\nu)^+}(\mathbb{R}_v^3)}\|g\|_{H^{m+\nu}_{(l+\gamma+\nu)^+}(\mathbb{R}_v^3)}\,. \tag{2.5}$$

Next we state the lower bound for the collision operator.

Lemma 2.1 (cf., [2, 11]). *Let $B = \Phi(|v-v_*|)b(\cos\theta)$ and let $\Phi = \langle v-v_*\rangle^\gamma$ with $\gamma \le 1$. Let b satisfy (1.7) or (1.9). Assume that $g \ge 0$, $\not\equiv 0$, $g \in L^1_{\max\{\gamma^+, 2-\gamma^+\}} \bigcap L\log L(\mathbb{R}_v^3)$. Then there exist constants $C_g > 0$ depending only on b, $\|g\|_{L^1_1}$ and $\|g\|_{L\log L}$ and $C > 0$ depending on b such that for any smooth function $f \in H^1_{\gamma/2}(\mathbb{R}_v^3) \cap L^2_{\gamma^+/2}(\mathbb{R}_v^3)$, we have*

$$-\Big(Q(g,\, f),\, f\Big)_{L^2(\mathbb{R}_v^3)} + C\|g\|_{L^1_{\max\{\gamma^+,\, 2-\gamma^+\}}(\mathbb{R}_v^3)}\|f\|^2_{L^2_{\gamma^+/2}(\mathbb{R}_v^3)}\,\cdot$$

$$\ge C_g \left\{ \begin{array}{ll} \|W_{\gamma/2}f\|^2_{H^{\nu/2}(\mathbb{R}_v^3)} & \textit{if (1.7) is satisfied,} \\ \|(\log\Lambda)^{(\mu+1)/2}W_{\gamma/2}f\|^2_{L^2(\mathbb{R}_v^3)} & \textit{if (1.9) is satisfied.} \end{array}\right. \tag{2.6}$$

Here

$$\|g\|_{L\log L} = \int_{\mathbb{R}^n} |g(v)|\log(1+|g(v|)dv.$$

Remark 2.2 *The factor $W_{\gamma/2}$ is crucial to show Theorem 1.1.*

Outline of proof. First, we have

$$\begin{aligned}
&(Q(g,f),f)\\
&= \int_{\mathbb{R}^6}\int_{\mathbf{S}^2} \Phi(|v-v_*|)b(\cos\theta)g(v_*)f(v)\{f(v')-f(v)\}d\sigma dv_* dv\\
&= \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbf{S}^2} \Phi(|v-v_*|)b(\cos\theta)g(v_*)\{f(v')^2-f(v)^2\}d\sigma dv_* dv\\
&- \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbf{S}^2} \Phi(|v-v_*|)b(\cos\theta)g(v_*)\{f(v')-f(v)\}^2 d\sigma dv_* dv\\
&= \mathcal{R}_1 - \mathcal{R}_2.
\end{aligned}$$

For $\mathcal{R}_1$, the change of the variable $v' \to v$ (see the cancellation lemma (Corollary 2 of [2]) we have

$$\begin{aligned}
\mathcal{R}_1 &= \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbf{S}^2} \Phi(|v-v_*|)b(\cos\theta)g(v_*)\{f(v')^2-f(v)^2\}d\sigma dv_* dv\\
&= \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbf{S}^2} \{\Phi(\frac{|v-v_*|}{\cos\frac{\theta}{2}})\frac{1}{\cos^3\frac{\theta}{2}} - \Phi(|v-v_*|)\}b(\cos\theta)g(v_*)f(v)^2 dv d\sigma dv_*\\
&= \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbf{S}^2} \Phi(\frac{|v-v_*|}{\cos\frac{\theta}{2}})\{\frac{1}{\cos^3\frac{\theta}{2}} - 1\}b(\cos\theta)g(v_*)f(v)^2 dv d\sigma dv_*\\
&+ \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbf{S}^2} \{\Phi(\frac{|v-v_*|}{\cos\frac{\theta}{2}}) - \Phi(|v-v_*|)\}b(\cos\theta)g(v_*)f(v)^2 dv d\sigma dv_*\\
&= \mathcal{R}_{11} + \mathcal{R}_{12}.
\end{aligned}$$

For the first term $\mathcal{R}_{11}$, from $1-\cos^3\frac{\theta}{2} \le 3(1-\cos\frac{\theta}{2}) = 6\sin^2\frac{\theta}{4}$, it follows that

$$\mathcal{R}_{11} \le C\|g\|_{L^1_{\gamma^+}}\|f\|^2_{L^2_{\gamma^+/2}},$$

because $\Phi \leq 1$ when $\gamma < 0$. For the second term $\mathcal{R}_{12}$, we first note that the mean value theorem gives

$$\begin{aligned}
&\Phi(\frac{|v-v_*|}{\cos\frac{\theta}{2}}) - \Phi(|v-v_*|) \\
&= -(\frac{1}{\cos\frac{\theta}{2}} - 1)|v-v_*|^2(1+(\frac{|v-v_*|}{a})^2)^{\frac{\gamma}{2}-1}\frac{2}{a^3} \\
&\leq C(\frac{1}{\cos\frac{\theta}{2}} - 1)\Phi(|v-v_*|),
\end{aligned}$$

where $\frac{\sqrt{2}}{2} \leq \cos\frac{\theta}{2} < a < 1$. Similarly to $\mathcal{R}_{11}$, we can obtain

$$\mathcal{R}_{12} \leq C\|g\|_{L^1_{\gamma^+}} \|f\|^2_{L^2_{\gamma^+/2}}.$$

Since $\mathcal{R}_2 \geq 0$ we have

$$\Big(Q(g,\, f),\, f\Big)_{L^2(\mathbb{R}^3_v)} \leq -C\|g\|_{L^1_{\gamma^+}(\mathbb{R}^3_v)}\|f\|^2_{L^2_{\gamma^+/2}(\mathbb{R}^3_v)}. \tag{2.7}$$

The further hard observation on $\mathcal{R}_2$ gives (2.6), (see Lemma 4.2 of [12]).

Here we newly give the commutator estimates between the collision operator Q and the moment weight W_l for $l \in \mathbb{N}$, though they are not given in [12].

Lemma 2.2 *Let $l \in \mathbb{N}$. Let $B = \Phi(|v-v_*|)b(\cos\theta)$ where $\Phi = \langle v - v_*\rangle^\gamma$ with $\gamma \leq 1$ and b satisfies (1.7). (1) When $0 < \nu < 1$, one has*

$$\begin{aligned}
&\left|\Big((W_l\, Q(f,\, g) - Q(f,\, W_l\, g)),\, h\Big)_{L^2(\mathbb{R}^3_v)}\right| \\
&\quad \leq C\|f\|_{L^1_{l+\gamma^+}(\mathbb{R}^3_v)}\|g\|_{L^2_{l+\gamma^+}(\mathbb{R}^3_v)}\|h\|_{L^2(\mathbb{R}^3_v)}.
\end{aligned} \tag{2.8}$$

(2) When $1 < \nu < 2$, for any $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that

$$\begin{aligned}
&\left|\Big((W_l\, Q(f,\, g) - Q(f,\, W_l\, g)),\, h\Big)_{L^2(\mathbb{R}^3_v)}\right| \\
&\quad \leq C_\varepsilon\|f\|_{L^1_{l+\nu-1+\gamma^+}(\mathbb{R}^3_v)}\|g\|_{H^{\nu-1+\varepsilon}_{l+\nu-1+\gamma^+}(\mathbb{R}^3_v)}\|h\|_{L^2(\mathbb{R}^3_v)}.
\end{aligned} \tag{2.9}$$

(3) When $\nu = 1$, we have the same estimate as (2.9) with $\nu - 1$ replaced by any small $\kappa > 0$.

Remark 2.3 *When $0 < \nu < 1$ and $l \geq 3 (> 5/2)$, the following variant of (2.8) holds*

$$\begin{aligned}
&\left|\Big((W_l\, Q(f,\, g) - Q(f,\, W_l\, g)),\, h\Big)_{L^2(\mathbb{R}^3)}\right| \\
&\quad \leq C\|f\|_{L^2_{l+\gamma^+}(\mathbb{R}^3_v)}\|g\|_{L^2_{l+\gamma^+}(\mathbb{R}^3)}\|h\|_{L^2(\mathbb{R}^3)}.
\end{aligned} \tag{2.10}$$

where the L^1 norm of f is replaced by its L^2 norm without increasing the weight.

As an application of upper-lower estimates and this remark, we consider the uniqueness of solution to the Cauchy problem (1.1).

Theorem 2.2 (cf. H.Tanaka[18], Toscani-Villani [19] for the case $\gamma = 0, \nu = \frac{1}{2}$) *Let $B = \Phi(|v - v_*|)b(\cos\theta)$ where $\Phi = \langle v - v_* \rangle^\gamma$ with $\gamma < 0$ and b satisfies (1.7) with $0 < \nu < 1$. Assume that $0 \le f, g \in C([0,T]; H^\nu_{l+(\gamma+\nu)^+})$ with $l \ge 3$. If f, g are solutions to the Cauchy problem (1.1) with the initial data $f_0 \in H^\nu_{l+(\gamma+\nu)^+}$ then they coincide.*

Proof. Setting $F = f - g$ and $G = f + g$, we have

$$\frac{\partial F}{\partial t} = \frac{1}{2}\left(Q(G,F) + Q(F,G)\right), \quad v \in \mathbb{R}^3, \ t > 0; \qquad F|_{t=0} = 0.$$

Multiply the equation by $W_{2l}F$ and integrate with respect to v variables in $\mathbb{R}^3$. Then we obtain

$$\frac{d\|F\|^2_{L^2_l}}{dt} = \Big(W_l Q(G,F), W_l F\Big) + \Big(W_l Q(F,G), W_l F\Big) = I_1 + I_2.$$

Write

$$I_1 = \Big(Q(G, W_l F), W_l F\Big) + \Big((W_l Q(G,F) - Q(G, W_l F)), W_l F\Big) = I_{1,1} + I_{1,2}.$$

By (2.7) we have

$$I_{1,1} \le C\|G\|_{L^1}\|F\|^2_{L^2_l}$$

and it follows from (2.10) we have

$$|I_{1,2}| \le C\|G\|_{L^2_l}\|F\|^2_{L^2_l}.$$

Write also

$$I_2 = \Big(Q(F, W_l G), W_l F\Big) + \Big((W_l Q(F,G) - Q(F, W_l G)), W_l F\Big) = I_{2,1} + I_{2,2}.$$

It follows from (2.3) with $m = 0$ that

$$|I_{2,1}| \le C||F||_{L^2_l}||G||_{H^\nu_{l+(\gamma+\nu)^+}}||F||_{L^2_l}$$

By (2.10) we have

$$|I_{2,2}| \le C||F||_{L^2_l}||G||_{L^2_l}||F||_{L^2_l}.$$

Summing up above estimates we obtain

$$\frac{d\|F\|^2_{L^2_l}}{dt} \le C_G\|F\|^2_{L^2_l},$$

which gives the uniqueness.

Remark 2.4 *By using the metric*

$$d_2(f,g) = \sup_{\xi \in \mathbb{R}^3} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^2},$$

Toscani-Villani [19] showed the uniqueness of solution in Maxwellian molecule case, assuming only finiteness of the energy, that is, $f, g \in L^1_2$.

Remark 2.5 *The proof of Theorem 2.2 can be applicable to the uniqueness of solutions to spatially inhomogeneous Boltzmann equation without angular cutoff in soft potential case (see [7]) where the hard potential case is also discussed.*

Theorem 2.2 uses the commutator estimates with respect to W_l. We also need the the following estimates concerning the commutator with respect to the Sobolev weight $|D_v|^2$. Typical example of such a weight is

$$M_\delta(D_v) = \frac{(1+|D_v|^2)^{N_0/2}}{(1+\delta|D_v|^2)^{N_1/2}}, \tag{2.11}$$

where $N_0, N_1 \in \mathbb{R}$ with $N_1 \geq N_0 + 4 > 0$, and $0 < \delta < 1$ is a parameter which tends to 0. It should be noted that the symbol $M_\delta(\xi)$ satisfies

$$|\partial_\xi^\alpha M_\delta(\xi)| \leq C_\alpha M_\delta(\xi)\langle\xi\rangle^{-|\alpha|}$$

for a constant C_α independent of δ.

Lemma 2.3 *Let $B = \Phi(|v-v_*|)b(\cos\theta)$ where $\Phi = \langle v-v_*\rangle^\gamma$ with $\gamma \leq 1$ and b satisfies (1.7). Let $\lambda \in \mathbb{R}$ and let $M(\xi)$ be a positive symbol of pseudo-differential operator in $S^\lambda_{1,0}$ of the form $M(\xi) = \tilde{M}(|\xi|^2)$. Assume that, there exist $c, C > 0$ such that*

$$c^{-1} \leq \frac{s}{\tau} \leq c \quad \textit{implies} \quad C^{-1}\frac{\tilde{M}(s)}{\tilde{M}(\tau)} \leq C \tag{2.12}$$

and $M(\xi)$ satisfies

$$|M^{(\alpha)}(\xi)| = |\partial_\xi^\alpha M(\xi)| \leq C_\alpha M(\xi)\langle\xi\rangle^{-|\alpha|} \tag{2.13}$$

for any α. Then, if $0 < \nu < 1$ then for any $N_1 \in \mathbb{N}$ there exist a constant C_{N_1} such that

$$\begin{aligned} &\left|(M(D_v)Q(f,g) - Q(f,\, M(D_v)g), h)_{L^2(\mathbb{R}^3_v)}\right| \\ &\leq C_{N_1}\|f\|_{L^1_{\gamma^+}(\mathbb{R}^3_v))}\Big(\|Mg\|_{L^2_{\gamma^+/2}(\mathbb{R}^3_v)} + \|g\|_{H^{\lambda-N_1}(\mathbb{R}^3_v)}\Big)\|h\|_{L^2_{\gamma^+/2}(\mathbb{R}^3_v)}. \end{aligned} \tag{2.14}$$

Furthermore, if $1 < \nu < 2$, for any $\varepsilon > 0$ and for any $N_1 \in \mathbb{N}$, there exists a constant C_{ε,N_1} such that

$$\begin{aligned} &\left|(M(D_v)Q(f,g) - Q(f,\, M(D_v)g), h)_{L^2(\mathbb{R}^3_v)}\right| \\ &\leq C_{\varepsilon,N_1}\|f\|_{L^1_{(\nu+\gamma-1)^+}(\mathbb{R}^3_v))}\Big(\|Mg\|_{H^{\nu-1+\varepsilon}_{(\nu+\gamma-1)^+}(\mathbb{R}^3_v)} + \|g\|_{H^{\lambda-N_1}(\mathbb{R}^3_v))}\Big)\|h\|_{L^2(\mathbb{R}^3_v)}\,. \end{aligned} \tag{2.15}$$

When $\nu = 1$ we have the same estimate as (2.15) with $(\nu+\gamma-1)$ replaced by $(\gamma+\kappa)$ for any small $\kappa > 0$.

Remark 2.6 *As stated in Lemma 5.1 of [12], we have the following better estimate in the case $1 < \nu < 2$*

$$\begin{aligned} &\left|(M(D_v)Q(f,g) - Q(f,\, M(D_v)g), h)_{L^2(\mathbb{R}^3_v)}\right| \\ &\leq C_{N_1}\|f\|_{L^1_{(\nu+\gamma-1)^+}(\mathbb{R}^3_v))}\Big(\|Mg\|_{H^{(\nu-1)/2}_{(\nu+\gamma-1)^+/2}(\mathbb{R}^3_v)} + \|g\|_{H^{\lambda-N_1}(\mathbb{R}^3_v))}\Big)\|h\|_{H^{(\nu-1)/2}_{(\nu+\gamma-1)^+/2}(\mathbb{R}^3_v)}\,. \end{aligned} \tag{2.16}$$

At the end of this section we give some elementary results derived from the usual pseudodifferential calculus.

Lemma 2.4 *Let p, r be in $\mathbb{R}$ and let $a(v), b(\xi) \in C^\infty$ satisfy for any $\alpha \in \mathbb{Z}_+^3$,*

$$\left|D_v^\alpha a(v)\right| \le C_{1,\alpha}\langle v\rangle^{r-|\alpha|}, \quad \left|\partial_\xi^\alpha b(\xi)\right| \le C_{2,\alpha}\langle \xi\rangle^{p-|\alpha|}$$

for some constants $C_{1,\alpha}, C_{2,\alpha} > 0$. Then there exists a constant $C > 0$ depending only on p, r and finite numbers of $C_{1,\alpha}, C_{2,\alpha} > 0$ such that for any $f \in \mathcal{S}(\mathbb{R}^3)$,

$$\begin{cases} ||a(v)b(D)f||_{L^2} \le C||\langle D\rangle^p \langle v\rangle^r f||_{L^2}, \\ \\ ||b(D)a(v)f||_{L^2} \le C||\langle v\rangle^r \langle D\rangle^p f||_{L^2}. \end{cases} \tag{2.17}$$

In particular, the two norms on the right hand sides of (2.17) *are equivalent to each other.*

Corollary 2.2 *Let $M_\delta(\xi)$ be given in* (2.11). *If $f \in L_2^1$, then there exists a constant $C_\delta > 0$ depending on $\delta > 0$ such that for any $\kappa \le 2$*

$$||\langle v\rangle^\kappa M_\delta(D_v)f||_{L^2} \le C_\delta ||f||_{L_2^1}.$$

Proof. Since $\left|M_\delta(\xi)^{(\alpha)}\right| \le C_{\delta,\alpha}\langle\xi\rangle^{-4-|\alpha|}$, it follows from (2.17) that

$$||\langle v\rangle^\kappa M_\delta(D_v)f||_{L^2} \le C_\delta ||\langle v\rangle^2 f||_{H^{-4}} \le C'_\delta ||\widehat{\langle\cdot\rangle^2 f}||_{L^\infty(\mathbb{R}^3_\xi)}.$$

It follows from Lemma 2.4 that the norm of H_l^k defined in (2.1) is equivalent to $||\langle v\rangle^l \langle D\rangle^k f||_{L^2}$. The following is a slight generalization of the interpolation estimates given in [11].

Lemma 2.5 *Let $p, r \in \mathbb{R}$ and $\varepsilon > 0$. Then, there exists a constant $C = C(p, r, \varepsilon) > 0$ such that for any $f \in \mathcal{S}(\mathbb{R}^3)$,*

$$||f||^2_{H_r^p} \le C||f||_{H_{2r}^{p-\varepsilon}} ||f||_{H^{p+\varepsilon}} \le C\big(||f||^2_{H_{2r}^{p-\varepsilon}} + ||f||^2_{H^{p+\varepsilon}}\big). \tag{2.18}$$

Proof. It follows from Lemma 2.4 that

$$\begin{aligned} ||\ f\ ||^2_{H_r^p} &= \big(\langle D\rangle^{-p-\varepsilon}\langle v\rangle^r \langle D\rangle^{2p}\langle v\rangle^r f, \langle D\rangle^{p+\varepsilon} f\big)_{L^2} \\ &\le ||\ \langle D\rangle^{-p-\varepsilon}\langle v\rangle^r \{\langle D\rangle^{2p}\langle v\rangle^r f\}\ ||_{L^2} ||\ f\ ||_{H^{p+\varepsilon}} \\ &\le C||\langle v\rangle^r \langle D\rangle^{-p-\varepsilon}\{\langle D\rangle^{2p}\langle v\rangle^r f\}\ ||_{L^2} ||\ f\ ||_{H^{p+\varepsilon}} \\ &\le C||\langle D\rangle^{p-\varepsilon}\langle v\rangle^r \{\langle v\rangle^r f\}\ ||_{L^2} ||\ f\ ||_{H^{p+\varepsilon}} \le C\ ||\ f\ ||_{H_{2r}^{p-\varepsilon}} ||\ f\ ||_{H^{p+\varepsilon}}\ . \end{aligned}$$

Proposition 2.1 *Let $k, r \in \mathbb{R}^+$ and $\varepsilon > 0$. If $\ell \in \mathbb{N}$ is bigger than $(k + 3/2)/\varepsilon$, then there exists a constant $C(k, r, \varepsilon) > 0$ such that for any $f \in \mathcal{S}(\mathbb{R}^3)$,*

$$||f||^2_{H_r^k} \le C(N, r, \varepsilon)\Big(||f||^2_{L^1_{r2^\ell}} + ||f||^2_{H^{k+\varepsilon}}\Big).$$

Proof. The repeated use of (2.18), ℓ times, yields

$$||f||^2_{H^k_r} \le C\Big(||f||^2_{H^{k-\varepsilon\ell}_{r2^\ell}} + ||f||^2_{H^{k+\varepsilon}}\Big).$$

Since $L^1 \subset H^{-m}$ if $m > 3/2$, we obtain the desired estimate.

Remark 2.7 *Since for any $\kappa > 0$ we have, instead of* (2.18),

$$||f||^2_{H^p_r} \le \kappa ||f||^2_{H^{p+\varepsilon}} + C_\kappa ||f||^2_{H^{p-\varepsilon}_{2r}}$$

and the symbols $M_\delta(\xi)$ belong to a bounded set of $S^{N_0}_{1,0}$ uniformly with respect to $0 < \delta < 1$, by means of Lemma 2.4 we have for a suitable large $r' > 0$

$$\begin{aligned} ||M_\delta f||^2_{H^k_r} &\le \kappa ||M_\delta f||^2_{H^{k+\varepsilon}} + C_\kappa ||\langle v\rangle^{r'}\{\langle D_v\rangle^{-2-N_0} M_\delta f\}||^2_{L^2} \\ &\le \kappa ||M_\delta f||^2_{H^{k+\varepsilon}} + C_\kappa ||\langle v\rangle^{r'} f||^2_{H^{-2}} \\ &\le \kappa ||M_\delta f||^2_{H^{k+\varepsilon}} + C_\kappa ||f||^2_{L^1_{r'}}, \end{aligned} \tag{2.19}$$

provided that $M_\delta f \in H^{k+\varepsilon}$ and $f \in L^1_{r'}$.

3 Sketch of Proofs of Theorems 1.1 and 1.2

We will first give the proof of Theorem 1.1. Before that, we give the precise definition of weak solution for the Cauchy problem (1.1), cf. [21].

Definition 3.1 *Let $f_0(v) \ge 0$ be a function defined on $\mathbb{R}^3$ with finite mass, energy and entropy. $f(t,v)$ is called a weak solution of the Cauchy problem (1.1), if it satisfies the following conditions:*

$$\begin{aligned} &f(t,v) \ge 0, \quad f(t,v) \in C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}^3)) \cap L^1([0,T]; L^1_{2+\gamma^+}(\mathbb{R}^3)), \\ &\quad f(0,v) = f_0(v), \\ &\int_{\mathbb{R}^3} f(t,v)\psi(v)dv = \int_{\mathbb{R}^3} f_0(v)\psi(v)dv \ \textit{for}\ \psi = 1, v_j, |v|^2; \\ &f(t,v) \in L^1 \log L^1, \quad \int_{\mathbb{R}^3} f(t,v)\log f(t,v)dv \le \int_{\mathbb{R}^3} f_0 \log f_0 dv, \quad \forall t \ge 0; \\ &\int_{\mathbb{R}^3} f(t,v)\varphi(t,v)dv - \int_{\mathbb{R}^3} f_0\varphi(0,v)dv - \int_0^t d\tau \int_{\mathbb{R}^3} f(\tau,v)\partial_\tau\varphi(\tau,v)dv \\ &\qquad\qquad = \int_0^t d\tau \int_{\mathbb{R}^3} Q(f,f)(\tau,v)\varphi(\tau,v)dv, \end{aligned} \tag{3.1}$$

where $\varphi(t,v) \in C^1(\mathbb{R}^+; C_0^\infty(\mathbb{R}^3))$. Here, the right hand side of the last integral given above is defined by

$$\begin{aligned} &\int_{\mathbb{R}^3} Q(f,f)(v)\varphi(v)dv \\ &\quad = \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} Bf(v_*)f(v)(\varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*))dvdv_*d\sigma. \end{aligned}$$

Hence, this integral is well defined for any test function $\varphi \in L^\infty([0,T]; W^{2,\infty}(\mathbb{R}^3))$ (see p. 291 of [21]).

For arbitrary large but fixed $k \in \mathbb{R}$, we take the time-dependent multiplier

$$M_\delta(\xi) = M_\delta(\xi;t) = \frac{(1+|\xi|^2)^{\frac{kt-4}{2}}}{(1+\delta|\xi|^2)^{\frac{kT+4}{2}}} \qquad \text{for } t \in [0,T].$$

Let f be a weak solution of the Cauchy problem (1.1). We know that $f(t) \in L^1(\mathbb{R}^3) \subset H^{-2}(\mathbb{R}^3)$ for all $t \in [0,T]$. Then, for any $\delta \in (0,1)$

$$M_\delta(D_v,t)f \in L^\infty([0,T_0]; W^{2,\infty}(\mathbb{R}^3)), \tag{3.2}$$

whose norm is bounded from above by $C_\delta \|f_0\|_{L^1}$.

We consider the Debye-Yukawa potential case in Theorem 1.1. Since for any $0 < \nu < 1$ we have

$$(\log \theta^{-1})^\mu \le C\theta^{-\nu} \qquad \text{for any } \theta \in (0,\pi/2],$$

it follows from (2.14) in Lemma 2.3 that

Lemma 3.1 *Under the hypothesis* (1.8)-(1.9) *for the Debye-Yukawa potential, we have*

$$|(Q(f,f), M_\delta^2 f) - (Q(f,M_\delta f), M_\delta f)| \le C\|f\|_{L^1_1}(\|M_\delta(D_v)f\|^2_{L^2_{1/2}} + \|f\|^2_{L^1}),$$

where the constant $C > 0$ is independent of $\delta \in (0,1)$.

Indeed, we set $M = M_\delta$ and $h = M_\delta f$ in (2.14). Setting $f = M_\delta f$ in Lemma 2.1, we have

Lemma 3.2 *Under the hypothesis* (1.8)-(1.9) *for the Debye-Yukawa potential, we have*

$$-(Q(f,M_\delta f), M_\delta f) \ge C_{f,1}\|(\log\Lambda)^{\frac{\mu+1}{2}}\langle\cdot\rangle^{\frac{1}{2}} M_\delta f\|^2_{L^2} - C_2\|\langle\cdot\rangle^{\frac{1}{2}} M_\delta f\|^2_{L^2},$$

where $\Lambda = (e+|D_v|^2)^{\frac12}$. Here constants $C_{f,1}, C_2 > 0$ depend only on b, $\|f\|_{L^1_1}$ and $\|f\|_{L\log L}$.

We take $M_\delta^2(D_v,t)f$ as a test function in the definition of the weak solution (3.1). In addition to (3.2), we have

$$M_\delta f \in C([0,T]; L^2(\mathbb{R}^3)), \tag{3.3}$$

and for any $t \in (0,T]$, we have

$$\begin{aligned} &\frac12 \int_{\mathbb{R}^3} f(t) M_\delta^2(t) f(t) dv - \frac12 \int_0^t \int_{\mathbb{R}^3} f(\tau)\Big(\partial_t M_\delta^2(\tau)\Big) f(\tau) dv d\tau \\ &= \frac12 \int_{\mathbb{R}^3} f_0 M_\delta^2(0) f_0 dv + \int_0^t \Big(Q(f,f)(\tau), M_\delta^2(\tau) f(\tau)\Big)_{L^2} d\tau, \end{aligned} \tag{3.4}$$

where $M_\delta(t)$ denotes $M_\delta(D_v;t)$. About the proof of (3.3) and (3.4) , we refer to the end of Section 3, [12].

Since

$$\partial_t M_\delta(\xi;t) = k\log\langle\xi\rangle M_\delta(\xi;t), \tag{3.5}$$

we obtain

$$\int_0^t \int_{\mathbb{R}^3} f(\tau)(\partial_\tau M_\delta^2(\tau)) f(\tau) dv d\tau \le 2k \int_0^t \|(\log\Lambda)^{1/2} M_\delta f(\tau)\|^2_{L^2} d\tau. \tag{3.6}$$

By combining Lemma 3.1, Lemma 3.2, (3.4), and (3.6), we then have

$$\begin{aligned}\|M_\delta(t)f(t)\|_{L^2}^2 &+ C_{f,1}\int_0^t \|(\log\Lambda)^{(\mu+1)/2}\langle\cdot\rangle^{\frac{1}{2}}M_\delta f(\tau)\|_{L^2}^2 d\tau\\ &\le \|M_\delta(0)f_0\|_{L^2}^2 + 2k\int_0^t \|(\log\Lambda)^{1/2}M_\delta f(\tau)\|_{L^2}^2 d\tau \qquad (3.7)\\ &+ C_{f,2}\int_0^t \|\langle\cdot\rangle^{\frac{1}{2}}M_\delta f(\tau)\|_{L^2}^2 d\tau + C_{f,3}\int_0^t \|f\|_{L^1}^2 d\tau.\end{aligned}$$

We now show that the terms $\|(\log\Lambda)^{1/2}\langle\cdot\rangle^{\frac{1}{2}}M_\delta f(\tau)\|_{L^2}$ and $\|\langle\cdot\rangle^{\frac{1}{2}}M_\delta f(\tau)\|_{L^2}$ can be controlled by

$$\|(\log\Lambda)^{(\mu+1)/2}\langle\cdot\rangle^{\frac{1}{2}}M_\delta f(\tau)\|_{L^2}^2.$$

Since $[(\log\Lambda)^{1/2}, \langle v\rangle^{-1/2}]$ is a L^2 bounded operator, for any $h \in H^{1/2}$ we have

$$\|(\log\Lambda)^{1/2}\langle\cdot\rangle^{-1/2}h\|_{L^2}^2 \le \|(\log\Lambda)^{1/2}h\|_{L^2}^2 + C\|h\|_{L^2}^2,$$

and moreover for any $\kappa > 0$ and any $m \in \mathbb{N}$ the estimate

$$\|(\log\Lambda)^{1/2}h\|_{L^2}^2 + \|h\|_{L^2}^2 \le \kappa\|(\log\Lambda)^{(\mu+1)/2}h\|_{L^2}^2 + C(\kappa,m)\|h\|_{H^{-m}}^2,$$

holds with a suitable $C(\kappa,m) > 0$. Putting $h = \langle\cdot\rangle^{\frac{1}{2}}M_\delta f$ we have $\|h\|_{H^{1/2}} \le C_\delta\|f\|_{L^1_{1/2}}$ by a similar way as in the proof of Corollary 2.2. Applying the above two estimates to (3.7) and taking m such that $m > kT$, we obtain

$$\|M_\delta(t)f(t)\|_{L^2}^2 \le \|M_\delta(0)f_0\|_{L^2}^2 + C_{f,k,3}\int_0^t \|f\|_{L^1_{\frac{1}{2}}}^2 d\tau, \qquad (3.8)$$

where we have used the fact that $\|\langle\cdot\rangle^{\frac{1}{2}}M_\delta f\|_{H^{-m}} \le C\|f\|_{L^1_{1/2}}$ for a $C > 0$ independent of δ. Note that $\|f\|_{L^1_{1/2}} \le \|f\|_{L^1_{\frac{1}{2}}} \le \|f_0\|_{L^1_{\frac{1}{2}}}$,

$$\|M_\delta(t)f(t)\|_{L^2}^2 = \|(1-\delta\triangle)^{-(\frac{kT+4}{2})}f(t)\|_{H^{kt-4}}^2,$$

and

$$\|M_\delta(0)f_0\|_{L^2}^2 = \|(1-\delta\triangle)^{-(\frac{kT+4}{2})}f_0\|_{H^{-4}}^2 \le \|f_0\|_{H^{-4}}^2 \le C\|f_0\|_{L^1}^2.$$

Then it follows from (3.8) that

$$\|(1-\delta\triangle)^{-(\frac{kT+4}{2})}f(t)\|_{H^{kt-4}}^2 \le C\|f_0\|_{L^1_{\frac{1}{2}}}^2,$$

where the constant $C > 0$ is independent of δ. Finally, for any given $t > 0$, since k can be chosen arbitrarily large, by letting $\delta \to 0$, we have $f(t) \in H^{+\infty}$. Now the proof of Theorem 1.1 for the Debye-Yukawa potential is completed.

The proof of Theorem 1.1 for the case $0 < \nu < 1$ and $0 \le \gamma \le 1$ is similar. Noting (2.14) in Lemma 2.3 and setting $g = M_\delta f$ in Lemma 2.1, we have

$$\begin{aligned}\|M_\delta(t)f(t)\|_{L^2}^2 &+ C_{f,1}\int_0^t \|\langle\cdot\rangle^{\frac{\gamma}{2}}M_\delta f(\tau)\|_{H^{\nu/2}}^2 d\tau\\ &\le \|M_\delta(0)f_0\|_{L^2}^2 + 2k\int_0^t \|(\log\Lambda)^{1/2}M_\delta f(\tau)\|_{L^2}^2 d\tau \qquad (3.9)\\ &+ C_{f,2}\int_0^t \|\langle\cdot\rangle^{\frac{\gamma}{2}}M_\delta f(\tau)\|_{L^2}^2 d\tau + C_{f,3}\int_0^t \|f\|_{L^1}^2 d\tau,\end{aligned}$$

so that for any given $t > 0$ we have $f(t) \in H^{+\infty}$ by the same way as in the previous proof for the Debye-Yukawa potential case. Now the proof of Theorem 1.1 is completed.

Finally, we shall prove Theorem 1.2. First, Theorem 1.1 implies, if combined with Proposition 2.1, that if the assumption (1.10) is fulfilled and if $0 < \nu < 1$ (with $\gamma \geq 0$) or for the Debye-Yukawa potential, any entropy solution is in H^k_r for any $k, r > 0$. This proves Theorem 1.2 for the case $0 < \nu < 1$(with $\gamma \geq 0$) and the Debye-Yukawa potential.

For the case $1 \leq \nu < 2$ or $\gamma < 0$, on the other hand, the above proof does not work unless extra estimates are available because the energy inequality (3.9) is to be replaced, in view of (2.15), by

$$\begin{aligned} &\|M_\delta(t)f(t)\|^2_{L^2} + C_{f,1}\int_0^t \|\langle\cdot\rangle^{\gamma/2}M_\delta f(\tau)\|^2_{H^{\nu/2}}d\tau \\ &\quad\leq \|M_\delta(0)f_0\|^2_{L^2} + 2k\int_0^t \|(\log\Lambda)^{1/2}M_\delta f(\tau)\|^2_{L^2}d\tau \\ &\quad + C_{f,2}\int_0^t \|\langle\cdot\rangle^{(\gamma+\nu-1)^+}M_\delta f(\tau)\|^2_{H^{\nu/2-\epsilon}}d\tau + C_{f,3}\int_0^t \|f\|^2_{L^1}d\tau, \end{aligned} \tag{3.10}$$

and since $\nu > 1$ or $\gamma < 0$, the term $\|\langle\cdot\rangle^{(\gamma+\nu-1)^+}M_\delta f(\tau)\|_{H^{\nu/2-\epsilon}}$ cannot be controlled by $\|\langle\cdot\rangle^{\gamma/2}\,M_\delta f(\tau)\|_{H^{\nu/2}}$. It is the assumption (1.10) that provides such estimates. Indeed, we can then use Proposition 2.1 or more precisely the estimates (2.19) in its remark which reduces (3.10) to

$$\|M_\delta(t)f(t)\|^2_{L^2} \leq \|M_\delta(0)f_0\|^2_{L^2} + C\int_0^t \|f\|^2_{L^1_{r'}}d\tau,$$

for a suitable large $r' > 0$. Thus, we can conclude that $f \in H^{+\infty}$ for $t > 0$. Clearly, the same conclusion holds for the case $\nu = 1$ in view of the last part of Lemma 2.3. Now the proof of Theorem 1.2 is also complete.

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